A New Look at Fluid Dynamics

Y. Nambu, Universty of Chicago andOsaka University

Newtonian dynamics is a mapping of a point to another point, string theory in its

original form is a mapping of a line segment to another line segment. Fuid dinamics

may be regarded as a mapping of 3-D onto itself.

Incompressibility may be imposed as a subsidiary condition on the Jacobian

for the change of volume elements. In this formulation, the multiplying factor in the

condition turns out to be the pressure.

$L = \int (mv^2/2 - V - \lambda([x, y, z)_{\xi\eta\zeta} - 1))d\xi d\eta d\zeta$

original coorinates $\overrightarrow{\xi}$, present coordinates $\overrightarrow{x}(\overrightarrow{\xi})$

$\lambda = \lambda(\xi)$ $[xyz]_{\xi\eta\zeta} = \partial(x, y, z)/\partial(\xi, \eta, \zeta)$

Equations of motion

$m\ddot{\overrightarrow{x}} : +\partial_x V + [\lambda, y, z]_{\xi\eta\zeta} = 0$

$[\lambda, y, z]_{\xi\eta\zeta} = [x, y, z]_{\xi\eta\zeta}[\lambda, y, z]_{xyz} = \partial_x\lambda$,etc.

First consider 2-D

$[x, y, z]- > [x, y|$

div $\overrightarrow{v} = 0$

This is a constraint of the first kind, *a la* Dirac, so to satisfy it automatically, write

$- > \overrightarrow{v} = (\partial\varphi, -\partial\varphi)$

vorticity $=-\Delta\varphi$

For steady stream $\partial_t\varphi = 0$, $\dot{x} = [x, \varphi]$ $\dot{y} = [y, \varphi]$

This has the same form as the Hamiltonian equation,with x, y being

equivalent to the canonical pair q, p,and$[x, y]$ is the Poisson bracket.

Formally one can quantize it.

This is an example of space quantization.

In 3-D one introduces two scalar fields φ, χ, and write

$\dot{x} = [x, \varphi, \chi]$.etc ($\varphi\overrightarrow{\nabla}\chi$ is called Clebsch potential.)

For n-D, introduce n-1 scalars, $\dot{O} = [O, \varphi, \chi, ...]$

This is an example of the Nambu's many Hamiltonian formalism.

The physical meaning of φ, χ, ..in n-D:. $\varphi = constant$ is a (n-1)-D manifold.

The intersection of all $\varphi's$ is a point.

Further topics

Compressible fluids

Additive set of compatible stream functions in 2-D

vortex ensemble
Speculations on Titius-Bode's law and space quantization

References
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J. Cresson(Laboratoire de Mathematique de Pau, Universite de Pau)
The stochastic hypothesis and the spacing of planetary systems

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Newtonian dynamics is a mapping of a point to another point, string theory in its original form is a mapping of a line segment to another line segment. Fluid dinamics may be regarded as a mapping of 3-D onto itself.
Incompressibility may be imposed as a subsidiary condition on the Jacobian for the change of volume elements. In this formulation, the multiplying factor in the condition turns out to be the pressure.

$L = \int (mv^2/2 - V - \lambda([x, y, z)_{\xi\eta\zeta} - 1))d\xi d\eta d\zeta$
original coorinates $\overrightarrow{\xi}$, present coordinates $\overrightarrow{x}(\overrightarrow{\xi})$
$\lambda = \lambda(\overrightarrow{\xi})$ $[xyz]_{\xi\eta\zeta} == \partial(x, y, z)/\partial(\xi, \eta, \zeta)$

Equations of motion From $\delta x, \delta y, \delta z$.and integrating by parts,
$m\ddot{\overrightarrow{x}} : +\partial_x V + [\lambda, y, z]_{\xi\eta\zeta} = 0$ $[\lambda, y, z]_{\xi\eta\zeta} = [x, y, z]_{\xi\eta\zeta}[\lambda, y, z]_{xyz} = \partial_x\lambda$,etc.

First consider 2-D
$[x, y, z]- > [x, y|$ div $\overrightarrow{v} = 0$
The constraint is of the first kind (*a la*Dirac), so to satisfy it automatically,

introduce a scalar field φ, and write
$- > \overrightarrow{v} == (\partial\varphi, -\partial\varphi)$
vorticity $=\partial_x v_y - \partial_y v_x = -\Delta\varphi$

For steady stream $\partial_t\varphi = 0$,
$\dot{x} = [x, \varphi]$ $\dot{y} = [y, \varphi]$
This has the same form as the Hamiltonian equation.

x, y is equivalent to the canonical pair q, p,and$[x, y]$ is the Poisson bracket,
φ is the Hamiltonian.
So formally one can quantize it.
This is an example of *space quantization.*
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In 3-D one introduces two scalar fields φ, χ, and write
$\dot{x} = [x, \varphi, \chi]$.etc $(\varphi\overrightarrow{\nabla}\chi$ is called Clebsch potential.$\overrightarrow{C}$, curl $\overrightarrow{C}=\overrightarrow{v}$
For n-D, introduce n-1 scalars, $\dot{O} = [O, \varphi, \chi, ...]$
This is an example of the Nambu's many Hamiltonian formalism.

The physical meaning of $\varphi, \chi, ..$In n-D:.$\varphi = const$ is a (n-1)D maifold.
The intersection of all $\varphi's$ is a 1D vector, the velocity.
Many Hamiltonian formalism (in a more general way) seems to be used in fluid
dynamcs.

In point dynamics, motion is determined when the force is given. In fluid dynamics,
the pressure is not *a priori* known. The motion of the fluid determines the pressure.
But the motion of the individual particles cannot be arbitrary in order to satisfy the
constraint. Since the accleration $\overrightarrow{\dot{v}}$ is equal to the gradient of a scalar, it must
satisfy curl $\overrightarrow{\dot{v}} = 0$.

For 2D, $(D/Dt)\Delta\varphi = (\partial/\partial t)\Delta\varphi + [\Delta\varphi, \varphi] = 0$
and from $[x, y] = const$, $[\ddot{x}, y] + [x, \ddot{y}] = -2[\dot{x}, \dot{y}]$

Thus for steady flow, $[\Delta\varphi, \varphi] = 0$, $\underline{- > \Delta\varphi = F(\phi)}$
If $F = 0$. $\Delta\varphi = 0, \varphi$ is conformal..

A uniformly rotating fluid (as seen from a rotating frame) is described by
$\Delta\varphi = c$, $\varphi = cr^2/2$,

For 3D the condition on $\varphi, \dot{\chi}$ is similar.With the Clebsch potential $\overrightarrow{C}$,
$v = \nabla \times C$, $\dot{v} = (\nabla \times \dot{C}) = [((\nabla \times C), \varphi.\chi]$
The condition becomes $0 = \nabla \times \dot{v} = \nabla\times(\nabla \times \dot{C}) = \Delta\dot{C}$,

Since $C = \nabla\varphi \times \nabla\chi$,
$\underline{\Delta(\nabla\varphi \times \nabla\chi) = F(\varphi, \chi)}$,

Vorticity $\overrightarrow{\chi} =, \Delta(\nabla\varphi \times\nabla\chi)$ is a conserved quantity(along the stream).

Example1, in 2D.$\varphi = c \ln r$. . $|v|$ =c/$|r|$.φ is the 2D Coulomb potential. This is a vortex around the origin. Positive c corresponds to anti-clocwise motion.The singularity at the origin can be interptreted as the core of the vortex or a point boundary of the fluid which does not move.

If φ is Yukawa-like,, $F = -m^2\varphi$,$\varphi = j_0(mr)$, a Bessel function.

Example2 $\varphi = cy$ steady horizontal flow with velocity c

Example 3 $\varphi = v \sin kx \exp(ky) - y + c$ horizontal wave pattern with decreacing amplitude downwards. There must be a surface where pressure is 0.

An additive set of compatible vortices.

Can one add two φs and still satisfy the compatibility?

$$\Delta(\varphi_1 + \varphi_2) = F(\varphi_1 + \varphi_2)?$$

This wi[redacted] be the case if $F = 0$,or $F = m^2\varphi$ with the same m.

For example, one can talk about an ensemble of vortices

$$\varphi = \Sigma c_n \ln |r - r_n|,$$
$$\varphi = \Sigma c_n j_0(m|r - r_n|)$$

The velocity at each pint is the sum of contributions from all vortex cores, The cores themselves are fixed in space. If they move with the stream, it is not a steady motion but it is permissible. For example, consider two vortices of equal vorticity. They circle about each other; if the pair is equal and opposite; the two will move together on a strait line perpendicular to the pair.

Relativistic form

The kinetic energy density in the action is replaced by

$mc^2 ds = -mc^2\sqrt{1 - v^2/c^2} dt$, The Jacobian suffers a Lorentz contraction$\sqrt{1 - v^2/c^2}$,so

$$L = -(mc^2 + \lambda)\sqrt{1 - v^2/c^2} + \lambda[x, y, z]$$

Thus the pressure $\lambda$is a Lorentz scalar. If the system is at rest in a frame, with an external vector field balancing against the pressure, in another frame the Lorentz transformed vector will balance the same λ.It also means that fluid with matter density m acquires an addtional mass λ/c^2 The equation of state in an arbitrary frame becomes $pV = RT\sqrt{1 - v^2/c^2}$

A consequence of this is the following. A matter in gas and liquid phase with the same density at the same temperature (say at the transition point) should have different weights. In a gas, there is the pressure term λ/c^2. For water at 100■ C this is a 10^{-13} correction.

Compressible fluids

If the fluid is compressible, density depends on pressure, for example $pV = kT$, or $pV^\gamma = const$. So the volume factor $[x, y, z]$■V depends on λ. In this way the equations of motion become nonlinear functions of λ,as does the compatbility condition

For $pV = kT$,or $[x, y, z] = kT/\lambda$

$$L = \frac{1}{2}v^2 + \lambda([x, y, z]_{\xi\eta\zeta} - -kT \ln \lambda,$$

$[\lambda, y, z] = \partial_x \lambda / kT\lambda = \partial_x(\ln \lambda)/kT$

2D quantization and Titius-Bode law.

As mentioned above, the 2D equation of motion $\dot{O} = [O, \varphi\backslash]_{xy}$ has the same form as the canonical equation $\dot{O} = [O, H]_{pq}., x, y$ being equivalent to p, q.

A difference in interpretation is that the variables in H are the canonical coordinates of a single particle whereas they are the actual values for the collection of particles

But it naturally suggests "quantization"with some "Planck's constant k".

So $r^2 {=} x^2 + y^2 = 2kn$ when quantized.

One is immediately reminded of the celebrated Titius-Bode law, more commonly known as the Bode's law regarding the regularity of the planetary orbits around the sun The semi-major axes R_n of the planets are given by

$$R_n = 0.4 + 0.3 \times 2^n, n = -\infty, 0.1, 2. ..$$

taken the earth's value as the unit. The moons around a planet also follow a similar pattern. A generic form can be expressed as

$$r_n = c\exp(kn)$$

The facors of 2 for the solar system suggest a bifurcation process,

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but it is hard to imagine such a process in the formation of planets.. So I will start

from the following picture.

First consider a collection (or gas) of planets around a central "sun" orbiting around it. Assuming circular orbits for simplicity, the accleration is v^2/r,the attractive force is GMm/r^2.so $v = \sqrt{GM/r}$,the angular velocity $\omega = \sqrt{GMr}$,the

stream function is $\varphi = 2\sqrt{GMr}$

Next consider a compressible gas of planets at rest around a sun. Its pressure p is

balanced against the central force and the attraction generated by other constituents. An equation of state for the gas determines the density ρ which in

turn produces a gravitational potential $-G\rho$. For an isothermal relation $pV = kT$,

the equation is given by

$$kT\rho(r) = GM/r + Gm \int (\rho(r)/|r - r^{'}|)(dr')^3$$

$$\Delta\rho(r) = -4\pi\delta(r) - (4\pi Gm/kT)\rho(R)$$

$$- > \rho = \exp(-4\pi GmkT\rho(r) + GM/(kTr)$$

Now let us give it a spin as a whole with angular velocity Ω around the z axis. The

centrifugal force $-m\rho r\Omega^2$ will reducepressure and density, and the system

will

flatten An equation of state for the gas determines the density ρ which in turn

produces a gravitational potential $-G\rho$.

For an isothermal relation $pV = kT$, the equation is given by

$$p(r) = kT\rho = GM/r + Gm \int \rho/[\overrightarrow{R} - \overrightarrow{r}[)(dR)^3 - \frac{1}{2}Gm\rho(x^2 + y^2)\Omega^2$$

$$\Delta[kT + \frac{1}{2}Gm(x^2 + y^2)\Omega)\rho] = -4\pi GM\delta^3(r)$$

Writing $kT + \frac{1}{2}Gm(x^2 + y^2)\Omega) = X$.

$\Delta(\rho X) = -Gm\rho - GM\delta^3(r) = (Gm\rho X)X^{-1}$

$X^{-1}\tilde{\ } \frac{1}{kT} - \frac{1}{2}Gm(x^2 + y^2)\Omega)/(kT)^2$

The solution is

$\rho\ \tilde{\ } C \exp[-\frac{1}{2}Gm(x^2 + y^2)\Omega](\frac{1}{kT} - \frac{1}{2}Gm(x^2 + y^2)\Omega)/(kT)^2)$

where the medium is effectively regarded as 2D.

Now the quasi-quantization gives $x^2 + y^2 = 2kn$, so the volume $1/\rho$ takes the form

of the Bode's law.essentially,

The numerical values are estimated as

$\ln 2 = 0.693\,15,$
$[x,y]_{\xi\eta} = F_\lambda = V = kT/p.$ $[\lambda, i]_{\xi\eta} = F'\partial_x\lambda = \partial_x F - > F = p$
->$F = kT/F$ $FF' = kT - > F^2 = 2kT\lambda$ $F = p = \sqrt{2kT\lambda}$

gravity G
earth accel g 10^3 $cm/\sec/\sec$
$= Gm/r^2$
earth radius $r = 6{*}10^5 cm$
earth vol $\frac{4\pi}{3}(6{*}10^5)^3 = 10^{18} cm^3$
earth mass $5 * 10^{18}$ gm
$- > g = Gm/r^2 = G * 5 * 10^{18}/4/10^{11}\tilde{} G * 10^7$
$- > G = 10^{-4}$
$Gm = 5 * 10^{14}$
$- > Gm\Omega/(kT)^2$=0.7
$\Omega = 5 * 10^{-8}$
$- > (kT)^2 = 3 * 10^7$ $kT\tilde{}3 * 10^3$
for ordinary matter 10^{-4}

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